STABILITY FOR SOME INVERSE PROBLEMS FOR TRANSPORT EQUATIONS

FIKRET GÖLGELEYEN[†] AND MASAHIRO YAMAMOTO[‡]

Abstract. In this article, we consider inverse problems of determining a source term and a coefficient of a first-order partial differential equation and prove conditional stability estimates with minimum boundary observation data and relaxed condition on the principal part.

Key words. Inverse Problem, Transport Equation, Stability Estimates.

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1. Introduction and main results. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and let $\nu(x)$ be the unit outward normal vector to $\partial\Omega$. Let us consider

$$\partial_t y(x,t) + H(x)\cdot\nabla y(x,t) + V(x)y(x,t) = f(x)R(x,t), \quad x \in \Omega,\, 0<t<T \tag{1.1}$$

and

$$y(x,0) = 0, \qquad x \in \Omega. \tag{1.2}$$

We assume that $H := (h_1, ..., h_n) \in \{C^1(\overline{\Omega})\}^n$ and $V \in L^\infty(\Omega)$.

Throughout this paper, we set $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $\partial_j = \frac{\partial}{\partial x_j}$ for $j = 1, 2, ..., n$ and $\partial_t = \frac{\partial}{\partial t}$, $\nabla = (\partial_1, ..., \partial_n)$, $\Delta = \sum_{j=1}^n \partial_j^2$, and $H\cdot J$ denotes the scalar product of $H, J \in \mathbb{R}^n$.

The main problems in this paper are

Inverse source problem

Let H, V, R, $\Gamma \subset \partial\Omega$, $T>0$ be given suitably. Determine $f(x)$, $x \in \Omega$ from $y|_{\Gamma\times(0,T)}$.

Moreover we consider

$$\partial_t u(x,t) + H(x)\cdot\nabla u(x,t) + V(x)u(x,t) = 0, \quad x\in\Omega,\, 0<t<T \tag{1.3}$$

and

$$u(x,0) = a(x), \qquad x \in \Omega. \tag{1.4}$$

Inverse coefficient problem

Let a and H be suitably given. Determine $V(x)$ and/or $H(x)$ by data $u|_{\Gamma\times(0,T)}$.

Equations (1.1) and (1.3) are transport equations and are models in physical phenomena such as Liouville equation and the mass conservation law. Moreover the transport equation is related to the integral geometry (e.g., Amirov [1]). As for other

[†]Department of Mathematics, Bulent Ecevit University, Zonguldak, 67100 Turkey (f.golgeleyen@beun.edu.tr). The work has been done during the stay of the first author at Graduate School of Mathematical Sciences of The University of Tokyo, which was supported by Leading Graduate Course for Frontiers of Mathematical Sciences and Physics.

[‡]Department of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo, 153-8914 Japan (myama@ms.u-tokyo.ac.jp). This author is partially supported by Grant-in-Aid for Scientific Research (S) 15H05740 of Japan Society for the Promotion of Science.

physical backgrounds such as neutron transport and medical tomography, see e.g., Case and Zweifel [8], Ren, Bal and Hielscher [18].

Our inverse problem is formulated with a single measurement, and Gaitan and Ouzzane [9], Klibanov and Pamyatnykh [15], Machida and Yamamoto [17] discuss the uniqueness and the stability for inverse problems for initial/boundary value problems for transport equations by Carleman estimates. The papers [15] and [17] discuss transport equations with integral terms where solutions y and u depend also on the velocity as well as the location x and the time t.

The main methodology in [9], [15], [17] is based on Bukhgeim and Klibanov [7]. After that, there have been many works. Limited to hyperbolic and parabolic equations, we can refer for example to Baudouin, de Buhan and Ervedoza [3], Beilina and Klibanov [4], Bellassoued and Yamamoto [6], Imanuvilov and Yamamoto [11], [12], Klibanov [14], Yamamoto [21] and the references therein. Here we do not intend to give any complete lists of the references. In [9] and [15], the key Carleman estimate is the same as the Carleman estimate for a second-order hyperbolic equation and in order to apply the Carleman estimate one has to extend the solutions y and u to (1.1) and (1.3) to the time interval $(-T, 0)$. Such an extension argument makes the proofs longer, and requires an extra condition to unknown coefficients and initial value as in [15]. In Sections 2 and 4, we prove Carleman estimates (Lemmata 1 and 3), which can directly estimate initial values. Thanks to our Carleman estimates, we can simplify the proofs of the stability and relax the constraints of the principal coefficients H's.

As for inverse problems for transport equations with many measurements, see surveys Bal [2], Stefanov [20] and the references therein. Klibanov and Yamamoto [16] established the exact controllability for the transport equations by a Carleman estimate.

Unlike [2] and [20], we discuss the inverse problems for a single initial/boundary value problem where we need not change initial values or boundary values. More precisely, in the formulation for the inverse problems in [2] and [20], we have to change boundary inputs on some subboundary and repeat measurements of the corresponding boundary outputs on other subboundary. One can apply the method of characteristics to the same kind of inverse problem for the first-order equation and see Belinskij [5], Chapter 5 of Romanov [19] for example.

We set

$$Q = \Omega \times (0, T)$$

and

$$\begin{cases} \partial\Omega_+ = \{x \in \partial\Omega;\, (\nu(x) \cdot H(x)) > 0\}, \\ \partial\Omega_- = \{x \in \partial\Omega;\, (\nu(x) \cdot H(x)) < 0\}. \end{cases}$$

Throughout this paper, we assume that $\psi \in C^2(\overline{\Omega})$ and $H = (h_1, ..., h_n) \in \{C^1(\overline{\Omega})\}^n$ satisfy

$$\mu := \min_{x \in \overline{\Omega}} (H(x) \cdot \nabla\psi(x)) > 0. \tag{1.5}$$

We here note by (1.5) that $|H(x)| \neq 0$ for $x \in \overline{\Omega}$.

We give four cases where (1.5) holds.

Case 1. We assume

$$|\nabla d| > 0 \quad \text{on } \overline{\Omega}, \qquad H(x) = \nabla d(x), \qquad x \in \overline{\Omega}$$

with some $d \in C^2(\overline{\Omega})$. Then (1.5) holds if we choose $\psi(x) = d(x)$, $x \in \Omega$.
Case 2. Let us assume that $\{(h_1(x), ..., h_n(x));\ x \in \overline{\Omega}\} \subset \mathbb{R}^n$ is separated from $(0, ..., 0)$ by a hyperplane $a_1x_1 + \cdots + a_nx_n = 0$ with some $a_1, ..., a_n \in \mathbb{R}$ and $|a_1| + \cdots + |a_n| \ne 0$. Then $\psi(x) = a_1x_1 + \cdots + a_nx_n$ or $\psi(x) = -a_1x_1 - \cdots - a_nx_n$ satisfies (1.5). In particular, (1.5) holds if $H(x)$ is a constant vector because $\{H(x); \overline{\Omega}\}$ is composed of one point. In fact, the separation condition means that $|(H(x)\cdot\nabla\psi(x))| = |a_1h_1(x) + \cdots + a_nh_n(x)| > 0$ for all $x \in \overline{\Omega}$ or < 0 for all $x \in \overline{\Omega}$.
Case 3. Let $0 \in \Omega$. We assume that there exists a constant $\delta_0 > 0$ such that

$$|H(x)| \ge \delta_0, \quad x \in \overline{\Omega}.$$

Then $\psi(x) = \sum_{j=1}^n x_jh_j(x)$ satisfies (1.5) if $\max_{x\in\overline{\Omega}} |x|$ is sufficiently small.
Proof. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&(H(x)\cdot\nabla\psi(x)) = \sum_{\ell=1}^n h_\ell(x)^2 + \sum_{\ell=1}^n h_\ell(x)\sum_{j=1}^n x_j\partial_\ell h_j(x)\\
\ge&\min_{x\in\overline{\Omega}} |H(x)|^2 - \left(\sum_{\ell=1}^n h_\ell(x)^2\right)^{\frac12}\left(\sum_{\ell=1}^n\left|\sum_{j=1}^n x_j\partial_\ell h_j(x)\right|^2\right)^{\frac12}\\
\ge&\delta_0^2 - \|H\|_{\{L^\infty(\Omega)\}^n}\left(\sum_{\ell=1}^n\left(\sum_{j=1}^n |x_j|^2\right)\left(\sum_{j=1}^n |\partial_\ell h_j(x)|^2\right)\right)^{\frac12}\\
\ge&\delta_0^2 - \max_{x\in\overline{\Omega}} |x|\|H\|_{\{L^\infty(\Omega)\}^n}\|\nabla H\|_{\{L^\infty(\Omega)\}^{n\times n}}.
\end{aligned}$$

Therefore (1.5) holds true if

$$0 < \max_{x\in\overline{\Omega}} |x| < \frac{\min_{x\in\overline{\Omega}} |H(x)|^2}{\|H\|_{\{L^\infty(\Omega)\}^n}\|\nabla H\|_{\{L^\infty(\Omega)\}^{n\times n}}}.$$

Case 4. We assume that there exists $i_0 \in \{1, 2, ..., n\}$ such that $h_{i_0}(x) > 0$ for all $x \in \overline{\Omega}$. Then we choose sufficiently small $b \in \mathbb{R}$ such that $\overline{\Omega} \subset \{(x_1, x_2, ..., x_n);\ x_{i_0} > b\}$. Setting $\psi(x) = (x_{i_0} - b)^2$, we verify that (1.5) holds. In fact,

$$(H(x)\cdot\nabla\psi(x)) = 2h_{i_0}(x)(x_{i_0} - b) > 0 \quad \text{for } x \in \overline{\Omega}.$$

Now we state the first main result concerning the stability for the inverse source problem.
Theorem 1
Let $y \in H^1(Q)$ satisfy (1.1) and (1.2), and let (1.5) be satisfied with some constant $\mu > 0$. We assume that

$$R(x, 0) \ne 0, \quad x \in \overline{\Omega} \tag{1.6}$$

and

$$\partial_t y, \partial_t R \in H^1(Q), \quad \partial_t R \in L^2(0, T; L^\infty(\Omega)).$$

Let

$$T > \frac{\max_{x\in\overline{\Omega}} \psi(x) - \min_{x\in\overline{\Omega}} \psi(x)}{\mu}. \tag{1.7}$$

(i) We assume

$$\|\partial_t y\|_{L^2(Q)} \le M$$

with fixed constant $M > 0$. Then there exist constants $\theta \in (0,1)$ and $C > 0$, which are dependent on $\Omega, T, H, \|V\|_{L^\infty(\Omega)}, \psi, M, \|\partial_t R\|_{L^2(0,T;L^\infty(\Omega))}$, such that

$$\|f\|_{L^2(\Omega)} \le C\left\{\left(\int_0^T \int_{\partial\Omega_+} (H\cdot\nu)|\partial_t y|^2 dS_x dt\right)^{\frac{\theta}{2}} + \left(\int_0^T \int_{\partial\Omega_+} (H\cdot\nu)|\partial_t y|^2 dS_x dt\right)^{\frac{1}{2}}\right\}$$

for all $f \in L^2(\Omega)$.

(ii) Without the assumption in (i) concerning $\|\partial_t y\|_{L^2(Q)}$, there exists a constant $C > 0$, which is dependent on $\Omega, T, H, \|V\|_{L^\infty(\Omega)}, \psi, \|\partial_t R\|_{L^2(0,T;L^\infty(\Omega))}$, such that

$$\|f\|_{L^2(\Omega)} \le C\left(\int_0^T \int_{\partial\Omega} |(H\cdot\nu)||\partial_t y|^2 dS_x dt\right)^{\frac{1}{2}}$$

for all $f \in L^2(\Omega)$.

(iii) In addition to (1.1) and (1.2), we assume

$$y = 0 \qquad \text{on } \partial\Omega_- \times (0,T). \tag{1.8}$$

Then there exists a constant $C > 0$, which is dependent on $\Omega, T, H, \|V\|_{L^\infty(\Omega)}$, ψ, $\|\partial_t R\|_{L^2(0,T;L^\infty(\Omega))}$, such that

$$C^{-1}\left(\int_0^T \int_{\partial\Omega_+} (H\cdot\nu)|\partial_t y|^2 dS_x dt\right)^{\frac{1}{2}} \le \|f\|_{L^2(\Omega)} \le C\left(\int_0^T \int_{\partial\Omega_+} (H\cdot\nu)|\partial_t y|^2 dS_x dt\right)^{\frac{1}{2}} \tag{1.9}$$

for all $f \in L^2(\Omega)$.

The conclusion of (i) is a stability estimate of Hölder type and holds under a priori boundedness $\|\partial_t y\|_{L^2(Q)} \le M$, which is called conditional stability. On the other hand, the conclusions of (ii) and (iii) are Lipschitz stability and in particular, with (1.8) we can have both-sided estimate (1.9) for our inverse problem.

We apply Theorem 1 to the inverse coefficient problem of determining $V(x)$.

Theorem 2

For $j = 1, 2$, let $u_j, \partial_t u_j \in H^1(Q)$ and let

$$\partial_t u_j + H(x)\cdot\nabla u_j + V_j(x)u_j(x,t) = 0 \quad \text{in } Q, \tag{1.10}$$

$$u_j(x,0) = a(x), \qquad x \in \Omega \tag{1.11}$$

and

$$u_j = h(x,t) \qquad \text{on } \partial\Omega_- \times (0,T) \tag{1.12}$$

with suitably given a and h. We assume that there exists $\psi \in C^2(\overline{\Omega})$ satisfying (1.5) for H, (1.7) holds and

$$u_j, \partial_t u_j \in H^1(Q) \cap L^2(0,T; L^\infty(\Omega)), \quad j = 1,2.$$

Moreover we assume

$$\|V_j\|_{L^\infty(\Omega)}, \|\partial_t u_j\|_{L^2(0,T;L^\infty(\Omega))} \le M, \quad j = 1,2 \tag{1.13}$$

and

$$|a| > 0 \qquad \text{on } \overline{\Omega}, \tag{1.14}$$

where $M > 0$ is arbitrarily fixed constant.

Then there exists a constant $C > 0$ depending on Ω, T, H, ψ, a, M such that

$$C^{-1}\|\partial_t(u_1 - u_2)\|_{L^2(\partial\Omega_+\times(0,T))} \le \|V_1 - V_2\|_{L^2(\Omega)} \le C\|\partial_t(u_1 - u_2)\|_{L^2(\partial\Omega_+\times(0,T))}. \tag{1.15}$$

Similarly to Theorem 2, we can discuss the determination of H, but we have to determine n functions as the components of H, and so repeats of measurements of boundary data after changing initial values suitably are necessary but arguments can be repeated similarly to the proof of Theorem 2. Here we omit detailed discussions for the determination of all the components of H, but we consider the determination of the potential in the case of potential flows. That is, we consider

$$\begin{cases} \partial_t \rho(x,t) + \operatorname{div}(\rho \nabla d(x)) = 0 \quad \text{in } Q, \\ \rho(x,0) = a(x), \qquad x \in \Omega. \end{cases} \tag{1.16}$$

The first-order partial differential equation in (1.16) describes the mass conservation under stationary potential flow $\nabla d(x)$. Then we discuss the inverse problem of determining the stationary potential d.

For the statement of the main result, we define an admissible set of unknown potentials d's. Let constants $M > 0$, $\delta_0 > 0$ and functions $g_1 \in C^2(\partial\Omega)$, $g_2 \in C^1(\partial\Omega)$ be arbitrarily chosen. We define the admissible set of d's by

$$\mathcal{D} = \mathcal{D}(\delta_0, M, g_1, g_2) := \{d \in C^2(\overline{\Omega});\ |\nabla d| \ge \delta_0 > 0 \text{ on } \overline{\Omega},$$

$$\|d\|_{C^2(\overline{\Omega})} \le M, \quad d|_{\partial\Omega} = g_1,\ \partial_\nu d|_{\partial\Omega} = g_2\}. \tag{1.17}$$

For the function g_2 given in (1.17), we set

$$\begin{cases} \Gamma_+ = \{x \in \partial\Omega;\ g_2(x) > 0\}, \\ \Gamma_- = \{x \in \partial\Omega;\ g_2(x) < 0\}. \end{cases} \tag{1.18}$$

We state our final main result.

Theorem 3

For $j = 1,2$, let $\rho_j, \partial_t\rho_j \in H^1(Q)$ and let

$$\partial_t \rho_j(x,t) + \operatorname{div}(\rho_j \nabla d_j(x)) = 0 \quad \text{in } Q \tag{1.19}$$

and

$$\rho_j = h \quad \text{on } \Gamma_- \times (0,T), \qquad \rho_j(x,0) = a(x), \quad x \in \Omega \tag{1.20}$$

with suitable a and h. We assume (1.14),

$$T > \frac{\sup_{d\in\mathcal{D}}(\max_{x\in\overline{\Omega}} d(x) - \min_{x\in\overline{\Omega}} d(x))}{\delta_0^2} \tag{1.21}$$

and

$$\left\{ \begin{array}{l} \partial_t\rho_j \in L^2(0,T;W^{1,\infty}(\Omega)) \cap H^1(Q), \\ \|\rho_j\|_{L^2(0,T;H^1(\Omega))} \le M, \quad j=1,2. \end{array} \right. \tag{1.22}$$

Then there exists a constant $C>0$ depending on $\Omega, T, a, \delta_0, M, g_1, g_2$ such that

$$C^{-1}\|\partial_t(\rho_1-\rho_2)\|_{L^2(\Gamma_+\times(0,T))} \le \|d_1-d_2\|_{H^2(\Omega)} \le C\|\partial_t(\rho_1-\rho_2)\|_{L^2(\Gamma_+\times(0,T))} \tag{1.23}$$

for all $d_1, d_2 \in \mathcal{D}$.

The proofs of Theorems 1-3 are based on an argument by the Carleman estimates, which was originated by Bukhgeim and Klibanov [7]. Here we used a modified argument by Imanuvilov and Yamamoto [11], [12] which discussed for inverse problems for second-order hyperbolic equations.

The paper is composed of four sections and an appendix. In Section 2, we prove a relevant Carleman estimate for the proofs of Theorems 1 and 2, and an energy estimate, and in Section 3, the proofs of Theorems 1 and 2 are completed. In Section 4, we prove another Carleman estimate for the proof of Theorem 3 whose weight function gives a Carleman estimate also for the Laplacian and complete the proof of Theorem 3. In Appendix, we prove the Carleman estimate for the Laplacian.

2. Key Carleman estimate and energy estimate. We recall that

$$Q = \Omega \times (0,T)$$

and we set

$$Pu = \partial_t u + H(x)\cdot\nabla u + V(x)u, \quad P_0 u = \partial_t u + H(x)\cdot\nabla u, \quad (x,t)\in Q,$$

$$M_0 = \beta\|\mathrm{div}\, H\|_{L^\infty(\Omega)}\|\mathrm{div}\,(H(H\cdot\nabla\psi))\|_{L^\infty(\Omega)}$$

and

$$\varphi(x,t) = -\beta t + \psi(x), \quad (x,t)\in Q \tag{2.1}$$

with $\psi \in C^2(\overline{\Omega})$ and $\beta > 0$, and

$$B(x) := \partial_t\varphi + (H\cdot\nabla\varphi) = -\beta + (H(x)\cdot\nabla\psi), \quad x\in\Omega. \tag{2.2}$$

First we prove

Lemma 1

(i) We have

$$\begin{aligned}
&s\int_\Omega B(x)|u(x,0)|^2 e^{2s\varphi(x,0)}dx + s^2\int_Q B^2(x)|u(x,t)|^2 e^{2s\varphi}dxdt\\
\le &2\int_Q |Pu|^2 e^{2s\varphi}dxdt + (sM_0 + 2\|V\|^2_{L^\infty(\Omega)})\int_Q |u(x,t)|^2 e^{2s\varphi}dxdt\\
+&s\int_0^T\int_{\partial\Omega} B(x)(\nu\cdot H)|u|^2 e^{2s\varphi}dS_x dt
\end{aligned}$$

for all $s > 0$ and $u \in H^1(Q)$ satisfying $u(\cdot,T) = 0$ in Ω.
(ii) We assume (1.5) and

$$0 < \beta < \mu := \min_{x\in\overline{\Omega}}(H(x)\cdot\nabla\psi(x)). \tag{2.3}$$

Then

$$s\int_\Omega |u(x,0)|^2 e^{2s\varphi(x,0)}dx + \frac{s^2(\mu-\beta)^2}{2}\int_Q |u(x,t)|^2 e^{2s\varphi}dxdt$$

$$\le 2\int_Q |Pu|^2 e^{2s\varphi}dxdt + s\int_0^T\int_{\partial\Omega_+} B(H\cdot\nu)|u|^2 e^{2s\varphi}dS_x dt \tag{2.4}$$

for all $s \ge s_0$ and $u \in H^1(Q)$ satisfying $u(\cdot,T) = 0$ in Ω, where

$$s_0 = \max\left\{\frac{4M_0}{(\mu-\beta)^2}, \frac{\sqrt{8}\|V\|_{L^\infty(\Omega)}}{\mu-\beta}\right\}.$$

Inequality (2.4) is an estimate of Carleman's type, which holds uniformly for sufficiently large $s > 0$. We emphasize that Carleman estimate (2.4) can estimate also the initial value $u(x,0)$, and the weight function is linear in t. In [17] such a linear weight function is used for proving a stability estimate for an inverse problem for a transport equation. For the transport equation, the works [9] and [15] used the same weight function as the second-order hyperbolic equation, that is, $\varphi(x,t) = e^{\lambda(d(x)-\beta t^2)}$. Our choice (2.1) of the weight function enables us not to need to take any extensions of u to $(0,-T)$ in discussing the inverse problem. In [15], the extension argument requires an extra assumption on the initial value and unknown coefficients in addition to (1.14).

Proof of Lemma 1 (i).
First we assume that $V = 0$. We set $w(x,t) = e^{s\varphi(x,t)}u(x,t)$ and $(Lw)(x,t) = e^{s\varphi(x,t)}P_0(e^{-s\varphi}w)$. Then

$$Lw = \{\partial_t w + (H(x)\cdot\nabla w)\} - sB(x)w.$$

Hence by $u(\cdot, T) = 0$, we have

$$
\begin{aligned}
&\int_Q |P_0 u|^2 e^{2s\varphi} dxdt = \int_Q |Lw|^2 dxdt \\
=&\int_Q |\partial_t w + (H\cdot \nabla w)|^2 dxdt + \int_Q |sB|^2 w^2 dxdt - 2s\int_Q Bw(\partial_t w + (H\cdot\nabla w))dxdt \\
\ge& -2s\int_Q B(\partial_t w + H\cdot\nabla w)wdxdt + s^2\int_Q B^2 w^2 dxdt \\
=& -s\int_Q (B\partial_t(w^2) + BH\cdot\nabla(w^2))dxdt + s^2\int_Q B^2w^2 dxdt \\
=& s\int_Q (\partial_t B + (\mathrm{div}\, BH))w^2 dxdt - s\int_0^T\int_{\partial\Omega} B(\nu\cdot H)w^2 dS_x dt \\
&+ s^2\int_Q B^2 w^2 dxdt + s\int_\Omega B(x)|w(x,0)|^2 dx \\
\ge& -M_0 s\int_Q w^2 dxdt - s\int_0^T\int_{\partial\Omega} B(\nu\cdot H)w^2 dS_x dt \\
&+ s^2\int_Q B^2 w^2 dxdt + s\int_\Omega B(x)|w(x,0)|^2 dx.
\end{aligned}
$$

Substituting $w = e^{s\varphi}u$, we have

$$
s\int_\Omega B(x)|u(x,0)|^2 e^{2s\varphi(x,0)} dx + s^2\int_Q B^2(x)|u|^2 e^{2s\varphi} dxdt
$$

$$
\le \int_Q |P_0 u|^2 e^{2s\varphi} dxdt + M_0 s\int_Q |u|^2 e^{2s\varphi} dxdt + s\int_0^T\int_{\partial\Omega} B(x)|(H\cdot\nu)||u|^2 e^{2s\varphi} dS_x dt. \tag{2.5}
$$

Next let $V \in L^\infty(\Omega)$, $\not\equiv 0$. Then

$$
\begin{aligned}
&|P_0 u|^2 = |P_0 u + Vu - Vu|^2 \le 2|P_0 u + Vu|^2 + 2|Vu|^2 \\
\le& 2|Pu|^2 + 2\|V\|^2_{L^\infty(\Omega)}|u|^2.
\end{aligned}
$$

Therefore (2.5) completes the proof of Lemma 1 (i).

Proof of Lemma 1 (ii).

By (2.2) and (2.3), we have $B(x) \ge \mu - \beta > 0$ on $\overline{\Omega}$. Therefore we absorb the second term on the right-hand side of the conclusion of Lemma 1 into the left-hand side. More precisely, we have

$$
s^2\int_Q B^2(x)|u|^2 e^{2s\varphi} dxdt \ge (\mu-\beta)^2 s^2\int_Q |u|^2 e^{2s\varphi} dxdt,
$$

and so we easily see that if $s \ge s_0$, then

$$
\begin{aligned}
&(\mu-\beta)^2 s^2 - M_0 s - 2\|V\|^2_{L^\infty(\Omega)} \\
\ge& \frac{(\mu-\beta)^2}{2}s^2 + \left(\frac{(\mu-\beta)^2}{4}s^2 - M_0 s\right) + \left(\frac{(\mu-\beta)^2}{4}s^2 - 2\|V\|^2_{L^\infty(\Omega)}\right) \\
\ge& \frac{(\mu-\beta)^2}{2}s^2.
\end{aligned}
$$

Thus, by noting

$$\int_0^T \int_{\partial\Omega} B(x)(H\cdot\nu)|u|^2 dS_x dt \le \int_0^T \int_{\partial\Omega_+} B(x)(H\cdot\nu)|u|^2 dS_x dt,$$

the proof of the part (ii) is completed.

Next we show the classical energy estimate.

Lemma 2.
Let $\|V\|_{L^\infty(\Omega)} \le M$ and $\|H\|_{\{C^1(\overline{\Omega})\}^n} \le M$ with arbitrarily fixed constant $M > 0$. Let $w \in H^1(Q)$ satisfy

$$\begin{cases} \partial_t w + H(x)\cdot\nabla w + Vw = F(x,t) & \text{in } Q, \\ w(x,0) = a(x), \qquad x \in \Omega. \end{cases} \tag{2.6}$$

Then there exists a constant $C > 0$, depending on Ω, T, M, such that

$$\int_\Omega |w(x,t)|^2 dx + \int_0^T \int_{\partial\Omega_+} (H\cdot\nu)|w|^2 dS_x dt$$

$$\le C\left(\|a\|^2_{L^2(\Omega)} + \|F\|^2_{L^2(Q)} + \int_0^T \int_{\partial\Omega_-} |(H\cdot\nu)||w|^2 dS_x dt\right), \quad 0 \le t \le T. \tag{2.7}$$

Proof.
Multiplying $\partial_t w + H\cdot\nabla w + Vw = F$ by $2w$ and integrating over Ω, we have

$$\partial_t \int_\Omega |w(x,t)|^2 dx + \sum_{\ell=1}^n \int_\Omega h_\ell \partial_\ell(|w|^2) dx + 2\int_\Omega V|w|^2 dx = 2\int_\Omega Fw dx.$$

Setting $E(t) = \int_\Omega |w(x,t)|^2 dx$ and integrating by parts, we obtain

$$E'(t) = -\int_{\partial\Omega} (H\cdot\nu)|w|^2 dS_x + \int_\Omega (\operatorname{div} H)|w|^2 dx$$
$$-2\int_\Omega Vw^2 dx + 2\int_\Omega Fw dx.$$

Therefore, noting that $2\int_\Omega |Fw|dx \le \int_\Omega |F|^2 dx + \int_\Omega |w|^2 dx$ and integrating over $(0,t)$, we have

$$E(t) - E(0) = -\int_0^t \left(\int_{\partial\Omega_+} + \int_{\partial\Omega_-}\right)(H\cdot\nu)|w|^2 dS_x dt + \int_0^t \int_\Omega (\operatorname{div} H)|w|^2 dx dt$$

$$-2\int_0^t \int_\Omega Vw^2 dx dt + \int_0^t \int_\Omega |F|^2 dx dt + \int_0^t E(\xi) d\xi.$$

Therefore

$$E(t) + \int_0^t \int_{\partial\Omega_+} (H\cdot\nu)|w|^2 dS_x dt \le \|a\|^2_{L^2(\Omega)} + \|F\|^2_{L^2(Q)}$$

$$+(2\|V\|_{L^\infty(\Omega)} + \|\mathrm{div}\, H\|_{L^\infty(\Omega)} + 1)\int_0^t E(\xi)d\xi + \left|\int_0^t \int_{\partial\Omega_-} (H\cdot\nu)|w|^2 dS_x dt\right| \quad (2.8)$$

for $0 \le t \le T$. Since $\int_0^t \int_{\partial\Omega_+}(H\cdot\nu)|w|^2 dS_x dt \ge 0$, $0 \le t \le T$, omitting the second term on the left-hand side of (2.8), we apply the Gronwall inequality, and we obtain

$$E(t) \le Ce^{CT}\Bigg(\|a\|^2_{L^2(\Omega)} + \|F\|^2_{L^2(Q)})$$

$$+\int_0^T \int_{\partial\Omega_-} |(H\cdot\nu)||w|^2 dS_x dt\Bigg), \quad 0 \le t \le T. \quad (2.9)$$

Substituting (2.9) into the third term on the right-hand side of (2.8), we complete the proof of the lemma.

3. Proofs of Theorems 1 and 2.

Proof of Theorem 1.

Henceforth $C > 0$ denotes generic constants which are independent of $s > 0$. We note that $Py = \partial_t y + H\cdot\nabla y + Vy$ and $\mu = \min_{x\in\overline{\Omega}}(H(x)\cdot\nabla\psi(x)) > 0$, and we set $R = \max_{x\in\overline{\Omega}}\psi(x)$ and $r = \min_{x\in\overline{\Omega}}\psi(x)$. By (1.7), we can choose $\beta > 0$ such that

$$T > \frac{R-r}{\beta}, \quad 0 < \beta < \mu. \quad (3.1)$$

In fact, it suffices to choose $\beta > 0$ such that $\beta \in (0,\mu)$ is sufficiently close to μ.

With this $\beta > 0$, we set

$$\varphi(x,t) = -\beta t + \psi(x), \qquad (x,t) \in Q. \quad (3.2)$$

Then (3.1) implies

$$\varphi(x,T) \le R - \beta T < r \le \varphi(x',0), \quad x, x' \in \overline{\Omega}.$$

Therefore by $\varphi \in C^1(\overline{Q})$, there exist $\delta_1 > 0$ and r_0, r_1 such that $R - \beta T < r_0 < r_1 < r$,

$$\begin{cases} \varphi(x,t) > r_1, & x \in \overline{\Omega},\ 0 \le t \le \delta_1, \\ \varphi(x,t) < r_0, & x \in \overline{\Omega},\ T - 2\delta_1 \le t \le T. \end{cases} \quad (3.3)$$

For applying Lemma 1, we need a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$ and

$$\chi(t) = \begin{cases} 1, & 0 \le t \le T - 2\delta_1, \\ 0, & T - \delta_1 \le t \le T. \end{cases} \quad (3.4)$$

We set

$$z = (\partial_t y)\chi.$$

Then $z(x,T) = 0$, $x \in \Omega$ and

$$Pz = \chi f(\partial_t R) + (\partial_t \chi)\partial_t y, \quad (x,t) \in Q$$

and

$$z(x,0) = f(x)R(x,0), \qquad x \in \Omega.$$

Applying Lemma 1 (ii) to z, we obtain

$$s\int_\Omega |z(x,0)|^2 e^{2s\varphi(x,0)}dx \le C\int_Q |\chi f(\partial_t R)|^2 e^{2s\varphi}dxdt + C\int_Q |(\partial_t\chi)\partial_t y|^2 e^{2s\varphi}dxdt + Ce^{Cs}D^2 \tag{3.5}$$

for all large $s > 0$. Here

$$D^2 = \int_0^T \int_{\partial\Omega_+} (H\cdot\nu)|\partial_t y|^2 dS_x dt.$$

Since $\partial_t\chi = 0$ for $0 \le t \le T - 2\delta_1$ or $T - \delta_1 \le t \le T$, by (3.3) and the a priori bound $\|\partial_t y\|_{L^2(Q)} \le M$, we have

$$\int_Q |(\partial_t\chi)\partial_t y|^2 e^{2s\varphi}dxdt = \int_{T-2\delta_1}^{T-\delta_1}\int_\Omega |(\partial_t\chi)\partial_t y|^2 e^{2s\varphi}dxdt \le Ce^{2sr_0}\int_{T-2\delta_1}^{T-\delta_1}\int_\Omega |\partial_t y|^2 dxdt \tag{3.6}$$

and

$$\int_Q |(\partial_t\chi)\partial_t y|^2 e^{2s\varphi}dxdt \le Ce^{2sr_0}M^2 \tag{3.7}$$

for all large $s > 0$. Moreover $R(x,0) \ne 0$ for $x \in \overline{\Omega}$ and $z(x,0) = f(x)R(x,0)$, $x \in \overline{\Omega}$, we have

$$\int_\Omega |z(x,0)|^2 e^{2s\varphi(x,0)}dx \ge C\int_\Omega |f(x)|^2 e^{2s\varphi(x,0)}dx.$$

Therefore (3.5) yields

$$s\int_\Omega |f(x)|^2 e^{2s\varphi(x,0)}dx \le C\int_Q |f(x)|^2 e^{2s\varphi(x,t)}dxdt + CM^2e^{2sr_0} + Ce^{Cs}D^2.$$

Since $\varphi(x,t) \le \varphi(x,0)$ for $(x,t) \in Q$, we have

$$\begin{aligned} s\int_\Omega |f(x)|^2 e^{2s\varphi(x,0)}dx &\le C\int_0^T\int_\Omega |f(x)|^2 e^{2s\varphi(x,0)}dxdt + CM^2e^{2sr_0} + Ce^{Cs}D^2 \\ &\le CT\int_\Omega |f(x)|^2 e^{2s\varphi(x,0)}dx + CM^2e^{2sr_0} + Ce^{Cs}D^2, \end{aligned}$$

that is,

$$(s - CT)\int_\Omega |f(x)|^2 e^{2s\varphi(x,0)}dx \le CM^2e^{2sr_0} + Ce^{Cs}D^2$$

for all large $s > 0$. Using $\varphi(x,0) > r_1$ by (3.3) and choosing $s > 0$ large, we obtain

$$se^{2sr_1}\int_\Omega |f(x)|^2 dx \le CM^2e^{2sr_0} + Ce^{Cs}D^2,$$

that is,

$$\|f\|^2_{L^2(\Omega)} \le CM^2 e^{-2sr_*} + Ce^{Cs}D^2 \tag{3.8}$$

for all large $s > s_*$, where $s_* > 0$ is a sufficiently large constant. Here we set $r_* := r_1 - r_0 > 0$. We separately consider the two cases: $D \ge M$ and $D < M$.
Case 1 $D \ge M$**:**
Estimate (3.8) implies

$$\|f\|^2_{L^2(\Omega)} \le (Ce^{-2sr_*} + Ce^{Cs})D^2. \tag{3.9}$$

Case 2 $D < M$**:**
Replacing C by Ce^{Cs_*}, we see that (3.8) holds for all $s > 0$. We make the right-hand side of (3.8) small in s. We choose $M^2e^{-2sr_*} = e^{Cs}D^2$, that is,

$$s = \frac{2}{C + 2r_*} \log \frac{M}{D}.$$

Therefore (3.8) is reduced to

$$\|f\|_{L^2(\Omega)} \le 2CM^{1-\theta}D^\theta,$$

where $\theta = \frac{2r_*}{C+2r_*} \in (0,1)$. Thus the proof of Theorem 1 (i) is completed.

Next we prove Theorem 1 (ii). We have

$$\begin{cases} \partial_t(\partial_t y) + H(x) \cdot \nabla \partial_t y + V\partial_t y = f(x)\partial_t R, & (x,t) \in Q, \\ (\partial_t y)(x,0) = f(x)R(x,0), & x \in \Omega. \end{cases}$$

Applying Lemma 2 to $\partial_t y$, we obtain

$$\int_\Omega |\partial_t y(x,t)|^2 dx + \int_0^T \int_{\partial\Omega_+} (H\cdot\nu)|\partial_t y|^2 dS_x dt$$

$$\le C\|f\|^2_{L^2(\Omega)} + C\int_0^T \int_{\partial\Omega_-} |(H\cdot\nu)||\partial_t y|^2 dS_x dt \tag{3.10}$$

for $0 \le t \le T$. Therefore, omitting the second term on the left-hand side of (3.10) and applying it to (3.6), we obtain

$$\int_Q |(\partial_t\chi)\partial_t y|^2 e^{2s\varphi} dxdt \le Ce^{2sr_0}\|f\|^2_{L^2(\Omega)} + Ce^{2sr_0}\int_0^T \int_{\partial\Omega_-} |(H\cdot\nu)||\partial_t y|^2 dS_x dt$$

and similarly to (3.8), from (3.5) we can obtain

$$\|f\|^2_{L^2(\Omega)} \le Ce^{-2sr_*}\|f\|^2_{L^2(\Omega)} + Ce^{Cs}\int_0^T \int_{\partial\Omega} |(H\cdot\nu)||\partial_t y|^2 dS_x dt.$$

Choosing $s > 0$ large, we can absorb the first term on the right-hand side into the left-hand side, and complete the proof of (ii).

Finally we prove (iii). By (1.8), the conclusion of (ii) immediately yields the second inequality in (1.9). Next, by (1.8) and (3.10), we have

$$\int_0^T \int_{\partial\Omega_+} |(H\cdot\nu)||\partial_t y|^2 dS_x dt \le C\|f\|^2_{L^2(\Omega)},$$

which proves the first inequality in (1.9). Thus the proof of Theorem 1 (iii) is completed.

Proof of Theorem 2.
Theorem 2 can be derived directly by Theorem 1. In fact, setting $y = u_1 - u_2$, $f = V_1 - V_2$ and $R = -u_2$, by (1.10) - (1.14) we have

$$\begin{cases} \partial_t y + H(x)\cdot\nabla y + V_1 y = f(x)R(x,t), \quad (x,t)\in Q, \\ y(x,0) = 0, \qquad x\in\Omega, \\ y|_{\partial\Omega_-\times(0,T)} = 0, \end{cases}$$

and $\partial_t y \in H^1(Q)$, $\|\partial_t R\|_{L^2(0,T;L^\infty(\Omega))} \le M$, $R(x,0) = -a(x) \ne 0$ for $x\in\overline{\Omega}$. Thus Theorem 1 (iii) yields the conclusion of Theorem 2, and the proof of Theorem 2 is completed.

4. Proof of Theorem 3.

We set

$$P_d u = \partial_t u + \nabla d\cdot\nabla u + u\Delta d.$$

For the proof, we further need a Carleman estimate for Δ with the same weight function for P_d. Unfortunately the weight function defined by (2.1) does not work as weight for a Carleman estimate for Δ. Thus we have to introduce a second large parameter in the weight function. That is, as the weight function, we set

$$\varphi_d(x,t) = e^{\lambda(-\beta t + d(x))}, \quad (x,t)\in Q, \tag{4.1}$$

where $\lambda > 0$ is chosen later. The weight function in the form (4.1) has been known as more flexible weight function producing a Carleman estimate (e.g., Hörmander [10], Isakov [13]).

In the existing works on inverse problems by Carleman estimates, given weight functions have been used, and to the best knowledge of the authors, Theorem 3 is the first case where a Carleman estimate with unknown coefficient as the weight, is seriously involved. The use of such a Carleman estimate is necessary in order that the admissible set $\mathcal{D}$ of unknown coefficients d's is generously formulated such as (1.17). Otherwise the admissible set of unknown coefficients should be more restrictive. For example, for inverse problems of determining principal parts for second-order hyperbolic equations, there are no works by Carleman estimate with weight function given by unknown coefficient, so that choices of admissible sets of unknown coefficients of the principal parts are very limited (e.g., Bellassoued and Yamamoto [6], Section 6 of Chapter 5). For it, we need a Carleman estimate where all the constants can be taken uniformly for arbitrary $d\in\mathcal{D}$.

We first show a Carleman estimate for P_d with weight (4.1), which should hold uniformly for all $d\in\mathcal{D}$. We fix $\beta, T > 0$ such that

$$0 < \beta < \delta_0^2, \quad T > \frac{\sup_{d\in\mathcal{D}}(\max_{x\in\overline{\Omega}} d(x) - \min_{x\in\overline{\Omega}} d(x))}{\beta}, \tag{4.2}$$

where the constant δ_0 characterizes $\mathcal{D}$ (see (1.17)). Henceforth we set

$$J_d(x,t) := \partial_t \varphi_d + (\nabla d \cdot \nabla \varphi_d) = \lambda \varphi_d(-\beta + |\nabla d|^2).$$

Lemma 3
For each $\lambda > 0$ there exists a constant $s_0 > 0$, which is dependent on λ, δ_0, M, such that we can choose a constant $C > 0$ satisfying

$$\int_\Omega s\lambda\varphi_d(x,0)|u(x,0)|^2 e^{2s\varphi_d(x,0)}dx + \int_Q s^2\lambda^2\varphi_d^2|u(x,t)|^2e^{2s\varphi_d}dxdt$$

$$\le C\int_Q |P_d u|^2 e^{2s\varphi}dxdt + \int_0^T\int_{\partial\Omega} sJ_d(\partial_\nu d)|u|^2 e^{2s\varphi_d}dS_x dt \tag{4.3}$$

for all $s \ge s_0$, all $d \in \mathcal{D}$ and all $u \in H^1(Q)$ satisfying $u(x,T) = 0$, $x \in \Omega$.

Here we note that the choices of C and s_0 in (4.3) are uniformly for $d \in \mathcal{D}$.
Proof.
The proof is similar to Lemma 1 by noting the independency of the constant of $d \in \mathcal{D}$. Let $d \in \mathcal{D}$ be chosen arbitrarily. First we consider the case of $P_d u = \partial_t u + \nabla d \cdot \nabla u$. We set $w = e^{s\varphi_d}u$, and

$$Lw = e^{s\varphi_d}P_d(e^{-s\varphi_d}w) = \partial_t w + \nabla d \cdot \nabla w - sJ_d(x,t)w.$$

We note $\partial_t J_d(x,t) = -\lambda^2\beta\varphi_d(-\beta + |\nabla d|^2)$, $\partial_k J_d(x,t) = \lambda^2\varphi_d(\partial_k d)(-\beta + |\nabla d|^2) + \lambda\varphi_d(\partial_k|\nabla d|^2)$ and

$$J_d(x,t) \ge \lambda\varphi_d(\delta_0^2 - \beta), \quad (x,t) \in Q,\ d \in \mathcal{D} \tag{4.4}$$

by (1.17). Then

$$\begin{aligned}
&\int_Q |P_d u|^2 e^{2s\varphi_d}dxdt = \int_Q |Lw|^2 dxdt \\
=&\int_Q |(\partial_t w + \nabla d \cdot \nabla w) - sJ_d w|^2 dxdt \\
=&\int_Q |\partial_t w + \nabla d \cdot \nabla w|^2 dxdt + s^2\int_Q J_d^2|w|^2dxdt - 2s\int_Q (J_d w)(\partial_t w + \nabla d \cdot \nabla w)dxdt \\
\ge& s^2\int_Q J_d^2|w|^2 dxdt - s\int_Q J_d(2w(\partial_t w))dxdt - s\int_Q J_d\nabla d \cdot (2w\nabla w)dxdt \\
=:& I_1 + I_2 + I_3.
\end{aligned}$$

Henceforth $C > 0$ denotes generic constants which are independent of $d \in \mathcal{D}$ and $s > 0$, but dependent on δ_0, M, λ. Next integration by parts yields

$$\begin{aligned}
I_2 =& -s\int_Q J_d\partial_t(|w|^2)dxdt = -s\int_\Omega [J_d|w|^2]_0^T dx + s\int_Q (\partial_t J_d)|w|^2 dxdt \\
\ge& s\int_\Omega \lambda\varphi_d(x,0)(-\beta + |\nabla d|^2)|w(x,0)|^2 dx - C\int_Q s\lambda^2\varphi_d|w|^2dxdt \\
\ge& s(\delta_0^2 - \beta)\int_\Omega s\lambda\varphi_d(x,0)|w(x,0)|^2dx - C\int_Q s\lambda^2\varphi_d|w|^2dxdt
\end{aligned}$$

and

$$\begin{aligned} I_3 &= -s\int_Q \sum_{k=1}^n (J_d\partial_k d)\partial_k(|w|^2)dxdt \\ &= -s\int_0^T \int_{\partial\Omega} \sum_{k=1}^n J_d(\partial_k d)\nu_k|w|^2 dS_x dt + s\int_Q \sum_{k=1}^n \partial_k(J_d\partial_k d)|w|^2 dxdt \\ &\ge -\int_0^T \int_{\partial\Omega} sJ_d(\partial_\nu d)|w|^2 dS_x dt - C\int_Q s\lambda^2\varphi_d|w|^2 dxdt. \end{aligned}$$

Hence

$$\begin{aligned} &\int_\Omega s\lambda\varphi_d(x,0)|w(x,0)|^2 dx + \int_Q s^2\lambda^2\varphi_d^2|w|^2 dxdt \\ \le\ & C\int_Q |Lw|^2 dxdt + C\int_Q s\lambda^2\varphi_d|w|^2 dxdt + \int_0^T\int_{\partial\Omega} sJ_d(\partial_\nu d)|w|^2 dS_x dt. \end{aligned} \tag{4.5}$$

We choose

$$s_0 > 2Ce^{\lambda(\beta T+M)}.$$

Then, since $\varphi_d(x,t) \ge e^{-\lambda\beta T - \lambda M}$ for $(x,t) \in Q$ and $d \in \mathcal{D}$, if $s \ge s_0$, then

$$\begin{aligned} &s^2\lambda^2\varphi_d^2 - Cs\lambda^2\varphi_d = s^2\lambda^2\varphi_d^2\left(1 - \frac{C}{s\varphi_d}\right) \\ \ge & s^2\lambda^2\varphi_d^2\left(1 - \frac{Ce^{\lambda(\beta T+M)}}{s_0}\right) \ge \frac{1}{2}s^2\lambda^2\varphi_d^2. \end{aligned}$$

Therefore (4.5) yields

$$\begin{aligned} &\int_\Omega s\lambda\varphi_d(x,0)|w(x,0)|^2 dx + \frac{1}{2}\int_Q s^2\lambda^2\varphi_d^2|w|^2 dxdt \\ \le\ & C\int_Q |Lw|^2 dxdt + \int_0^T\int_{\partial\Omega} sJ_d(\partial_\nu d)|w|^2 dS_x dt \end{aligned} \tag{4.6}$$

for all $s \ge s_0$. Henceforth $s_0 > 0$ denotes generic constants which are dependent on λ, δ_0, M. Since $w = ue^{s\varphi_d}$, we rewrite (4.6) in view of u, so that

$$\begin{aligned} &\int_\Omega s\lambda\varphi_d(x,0)|u(x,0)|^2 e^{2s\varphi_d(x,0)} dx + \int_Q s^2\lambda^2\varphi_d^2|u|^2 e^{2s\varphi_d} dxdt \\ \le\ & C\int_Q |P_d u|^2 e^{2s\varphi_d} dxdt + \int_0^T\int_{\partial\Omega} sJ_d(\partial_\nu d)|u|^2 e^{2s\varphi_d} dS_x dt \end{aligned}$$

for all $s \ge s_0$. Here we note that $C > 0$ and $s_0 > 0$ may vary line by line, but dependent only on λ, δ_0, M. Next

$$\begin{aligned} &|\partial_t u + \nabla d\cdot\nabla u + (\Delta d)u|^2 \le 2|\partial_t u + \nabla d\cdot\nabla u|^2 + 2|(\Delta d)u|^2 \\ \le & 2|\partial_t u + \nabla d\cdot\nabla u|^2 + 2M^2|u|^2 \end{aligned}$$

in Q for each $d \in \mathcal{D}$. Similarly to the proof of Lemma 1 (i), we can finish the proof of (4.3).

Next we show a Carleman estimate for Δ with the weight function $\varphi_d(x,0)$.
Lemma 4
There exists a constant $\lambda_0 = \lambda_0(\delta_0, M) > 0$ such that for any $\lambda \ge \lambda_0$, there exists a constant $s_1 = s_1(\lambda, \delta_0, M) > 0$ satisfying: we can choose $C > 0$ such that

$$\begin{aligned}&\int_\Omega (s\lambda^2\varphi_d(x,0)|\nabla f(x)|^2 + s^3\lambda^4\varphi_d(x,0)^3|f(x)|^2)e^{2s\varphi_d(x,0)}dx\\ \le& C\int_\Omega |\Delta f|^2 e^{2s\varphi_d(x,0)}dx\end{aligned}$$

for all $s \ge s_1$, $d \in \mathcal{D}, f \in H_0^2(\Omega)$.

Lemma 4 is a Carleman estimate for Δ and the Carleman estimate for Δ is well known (e.g., Hörmander [10], Isakov [13]). However we need the uniformity of the constants s_0, C with respect to $d \in \mathcal{D}$ in the Carleman estimate. Thus in Appendix we give a direct proof by integration by parts, which differs from [10], [13].

Now we proceed to
Proof of Theorem 3.
First Step. We set

$$m_0 = \sup_{d\in\mathcal{D}}(\max_{x\in\overline{\Omega}} d(x) - \min_{x\in\overline{\Omega}} d(x)).$$

By assumption (1.21): $T > \frac{m_0}{\delta_0^2}$ of the theorem, for arbitrary $d \in \mathcal{D}$, we can choose $\beta > 0$ and fix such that

$$0 < \beta < \delta_0^2, \qquad T > \frac{m_0}{\beta}. \tag{4.7}$$

We choose $\lambda_0 > 0$ given in Lemma 4 and fix. Next we set $s_* = \max\{s_0(\lambda_0, \delta_0, M), s_1(\lambda_0, \delta_0, M)\}$, where $s_j(\lambda_0, \delta_0, M)$, $j = 0, 1$ are proved to exist in Lemmata 3 and 4. Since $\partial_\nu d = g_2$ on $\partial\Omega$ for any $d \in \mathcal{D}$, we note that the subboundaries Γ_+ and Γ_- defined by (1.18) correspond to $\partial\Omega_+$ and $\partial\Omega_-$ respectively. Since $\lambda_0 > 0$ is fixed, henceforth by C denoting generic constants which are independent of $s \ge s_*$ and $d \in \mathcal{D}$ but dependent on λ_0, a, β, m_0, Ω, T, noting (4.7) and (4.4), by Lemmata 3 and 4 we obtain

$$\begin{aligned}&\int_\Omega s|y(x,0)|^2 e^{2s\varphi_d(x,0)}dx + \int_Q s^2|y|^2 e^{2s\varphi_d}dxdt\\ \le& C\int_Q |P_d y|^2 e^{2s\varphi_d}dxdt + Ce^{Cs}\int_0^T\int_{\Gamma_+}|y|^2 dS_x dt\end{aligned} \tag{4.8}$$

and

$$\int_\Omega (s|\nabla f|^2 + s^3|f|^2)e^{2s\varphi_d(x,0)}dx \le C\int_\Omega |\Delta f|^2 e^{2s\varphi_d(x,0)}dx \tag{4.9}$$

for all $s \ge s_*$, $d \in \mathcal{D}$, $y \in H^1(Q)$ satisfying $y(\cdot, T) = 0$ in Ω and $f \in H_0^2(\Omega)$. Thus we obtained two Carleman estimates which hold uniformly for arbitrary $d \in \mathcal{D}$.

Second Step. We set $y = \rho_1 - \rho_2$, $f = d_1 - d_2$ and $R(x,t) = -\rho_2(x,t)$ in Q. Then, by (1.19) and (1.20), we obtain

$$\begin{cases} \partial_t y + \nabla d_1 \cdot \nabla y + y\Delta d_1 = \nabla f \cdot \nabla R + R\Delta f \quad \text{in } Q, \\ y(x,0) = 0, \qquad x \in \Omega, \\ y = 0 \qquad \text{on } \Gamma_- \times (0,T). \end{cases}$$

We differentiate this partial differential equation with respect to y in t, noting that $\partial_t y \in H^1(Q)$, $\partial_t R \in L^2(0,T; W^{1,\infty}(\Omega))$ and $R(x,0) = -a(x)$. Then we have

$$\begin{cases} & \partial_t(\partial_t y) + \nabla d_1 \cdot \nabla(\partial_t y) + (\partial_t y)\Delta d_1 \\ = & \nabla f \cdot \nabla \partial_t R + (\partial_t R)\Delta f \quad \text{in } Q, \\ & \partial_t y(x,0) = -\nabla f \cdot \nabla a - a\Delta f, \quad x \in \Omega, \\ & \partial_t y = 0 \qquad \text{on } \Gamma_- \times (0,T). \end{cases} \tag{4.10}$$

Applying Lemma 2 to (4.10) and estimating $\partial_t y$, we readily verify the first inequality in (1.23). Thus the rest part of this section is devoted to the proof of the second inequality of (1.23).

Since $y(\cdot, T) = 0$ does not hold, for applying (4.8), we need a cut-off function. Noting

$$\varphi_{d_1}(x,0) = e^{\lambda_0 d_1(x)} \ge e^{\lambda_0 \min_{x\in\overline{\Omega}} d_1(x)}$$

and

$$\varphi_{d_1}(x,T) = e^{\lambda_0(-\beta T + d_1(x))} \le e^{-\lambda_0 \beta T} e^{\lambda_0 \max_{x\in\overline{\Omega}} d_1(x)}$$

for all $x \in \overline{\Omega}$. Henceforth we set

$$r_0 = \frac{1}{2}\lambda_0 e^{-\lambda_0\beta - \lambda_0 M}(\beta T - m_0).$$

By (4.7) we have $r_0 > 0$. We note that $r_0 > 0$ is independent of $s > 0$ and special choices of $d_1, d_2 \in \mathcal{D}$, but dependent on δ_0, M, λ_0.

The mean value theorem implies

$$\begin{aligned} &\max_{x\in\overline{\Omega}} \varphi_{d_1}(x,T) - \min_{x\in\overline{\Omega}} \varphi_{d_1}(x,0) = e^{-\lambda_0\beta T + \lambda_0 \max_{x\in\overline{\Omega}} d_1(x)} - e^{\lambda_0 \min_{x\in\overline{\Omega}} d_1(x)} \\ =& e^{\xi_1}\lambda_0\{(\max_{x\in\overline{\Omega}} d_1(x) - \min_{x\in\overline{\Omega}} d_1(x)) - \beta T\} \le \lambda_0 e^{\xi_1}(m_0 - \beta T), \end{aligned}$$

where ξ_1 is a number between $\lambda_0 \min_{x\in\overline{\Omega}} d_1(x)$ and $-\lambda_0\beta T + \lambda_0 \max_{x\in\overline{\Omega}} d_1(x)$.

Therefore, by $d \in \mathcal{D}$, we have $e^{\xi_1} \ge e^{-\lambda_0\beta T - \lambda_0 M}$, and

$$\max_{x\in\overline{\Omega}} \varphi_{d_1}(x,T) - \min_{x\in\overline{\Omega}} \varphi_{d_1}(x,0) \le -\lambda_0 e^{\xi_1}(\beta T - m_0) \le -\lambda_0 e^{-\lambda_0\beta T - \lambda_0 M}(\beta T - m_0) = -2r_0. \tag{4.11}$$

For any $d_1 \in \mathcal{D}$, we have

$$\begin{aligned} &|\max_{x\in\overline{\Omega}} \varphi_{d_1}(x,T) - \max_{x\in\overline{\Omega}} \varphi_{d_1}(x,t)| = |\exp(\lambda_0(-\beta T + \max_{x\in\overline{\Omega}} d_1(x))) - \exp(\lambda_0(-\beta t + \max_{x\in\overline{\Omega}} d_1(x))| \\ &= |\exp(\lambda_0 \max_{x\in\overline{\Omega}} d_1(x))||e^{-\lambda_0\beta T} - e^{-\lambda_0\beta t}| \le e^{\lambda_0 M}\lambda_0\beta|T - t|. \end{aligned} \tag{4.12}$$

At the last inequality, by $d_1 \in \mathcal{D}$ and the mean value theorem, we have

$$\max_{x\in\overline{\Omega}} d_1(x) \le M$$

and

$$|e^{-\lambda_0\beta T} - e^{-\lambda_0\beta t}| = |e^{-\xi_2}(-\lambda_0\beta T + \lambda_0\beta t)| \le \lambda_0\beta|T-t|,$$

where $\xi_2 \ge 0$ is some constant.

We fix a constant $\delta_1 > 0$ sufficiently small such that

$$2e^{\lambda_0 M}\lambda_0\beta\delta_1 < r_0.$$

Then, for $T - 2\delta_1 \le t \le T$ and any $d_1 \in \mathcal{D}$, by (4.11) and (4.12), we have

$$\begin{aligned}
&\max_{x\in\overline{\Omega}} \varphi_{d_1}(x,t) - \min_{x\in\Omega} \varphi_{d_1}(x,0)\\
=&\max_{x\in\overline{\Omega}} \varphi_{d_1}(x,t) - \max_{x\in\overline{\Omega}} \varphi_{d_1}(x,T) + \max_{x\in\overline{\Omega}} \varphi_{d_1}(x,T) - \min_{x\in\overline{\Omega}} \varphi_{d_1}(x,0)\\
\le&|\max_{x\in\overline{\Omega}} \varphi_{d_1}(x,t) - \max_{x\in\overline{\Omega}} \varphi_{d_1}(x,T)| - 2r_0 < e^{\lambda_0 M}\lambda_0\beta|t-T| - 2r_0\\
\le&2e^{\lambda_0 M}\lambda_0\beta\delta_1 - 2r_0 < r_0 - 2r_0 = -r_0.
\end{aligned}$$

Therefore

$$\max_{x\in\overline{\Omega}} \varphi_{d_1}(x,t) < \mu_0 - r_0, \quad T - 2\delta_1 \le t \le T, \tag{4.13}$$

where we set $\mu_0 := \sup_{d\in\mathcal{D}} \min_{x\in\Omega} e^{\lambda_0 d(x)}$.

Let $\chi \in C_0^\infty(\mathbb{R})$ satisfy $0 \le \chi \le 1$ and (3.4) with here chosen δ_1. We set

$$z = \chi\partial_t y.$$

Then, by (4.10), we have

$$\begin{cases}
& P_{d_1} z = \partial_t z + \nabla d_1 \cdot \nabla z + z\Delta d_1\\
= & \chi\nabla f \cdot \nabla\partial_t R + \chi(\partial_t R)\Delta f + (\partial_t\chi)\partial_t y \quad \text{in } Q,\\
& z(x,0) = -\nabla f\cdot\nabla a - a\Delta f, \quad z(x,T) = 0 \quad x \in \Omega,\\
& z = 0 \qquad \text{on } \Gamma_- \times (0,T).
\end{cases} \tag{4.14}$$

Applying (4.8) to (4.14), we obtain

$$\begin{aligned}
&\int_\Omega s|a\Delta f + \nabla f\cdot\nabla a|^2 e^{2s\varphi_{d_1}(x,0)}dx\\
\le\, & C\int_Q |\chi\nabla f\cdot\nabla\partial_t R + \chi(\partial_t R)\Delta f|^2 e^{2s\varphi_{d_1}}dxdt + C\int_Q |(\partial_t\chi)\partial_t y|^2 e^{2s\varphi_{d_1}}dxdt + Ce^{Cs}D^2
\end{aligned} \tag{4.15}$$

for all $s \ge s_*$. Here and henceforth we set

$$D^2 = \int_0^T \int_{\Gamma_+} |\partial_t(\rho_1 - \rho_2)|^2 dS_x dt.$$

Apply Lemma 2 to (4.10), by (1.21) we have

$$\int_\Omega |\partial_t y(x,t)|^2 dx \le C(\|\nabla f\cdot\nabla a + a\Delta f\|^2_{L^2(\Omega)} + \|\nabla f\cdot\nabla(\partial_t R) + (\partial_t R)\Delta f\|^2_{L^2(Q)}) + CD^2$$

$$\le C(\|\Delta f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{\{L^2(\Omega)\}^n} + D^2), \quad 0\le t\le T. \tag{4.16}$$

Consequently, since $\partial_t\chi \ne 0$ only if $T-2\delta_1 \le t \le T-\delta_1$ by (3.4), it follows from (1.14) that (4.15) and (4.16) yield

$$\begin{aligned}
&\int_\Omega s|\Delta f|^2 e^{2s\varphi_{d_1}(x,0)}dx - C\int_\Omega s|\nabla f|^2 e^{2s\varphi_{d_1}(x,0)}dx\\
\le& C\int_\Omega (|\Delta f|^2 + |\nabla f|^2)e^{2s\varphi_{d_1}(x,0)}dx\\
+& C\left(\int_{T-2\delta_1}^{T-\delta_1}\int_\Omega e^{2s\varphi_{d_1}}dxdt\right)(\|\Delta f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{\{L^2(\Omega)\}^n} + D^2) + Ce^{Cs}D^2
\end{aligned}$$

for all $s \ge s_*$. For the first integral on the right-hand side, we used $\varphi_{d_1}(x,t) \le \varphi_{d_1}(x,0)$ for $(x,t)\in Q$, and so

$$\begin{aligned}
&\int_Q |\chi\nabla f\cdot\nabla(\partial_t R) + \chi(\partial_t R)\Delta f|^2 e^{2s\varphi_{d_1}(x,t)}dxdt\\
\le& C\int_Q (|\nabla f|^2 + |\Delta f|^2)e^{2s\varphi_{d_1}(x,t)}dxdt \le C\int_\Omega (|\nabla f|^2 + |\Delta f|^2)e^{2s\varphi_{d_1}(x,0)}dx.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\int_\Omega s|\Delta f|^2 e^{2s\varphi_{d_1}(x,0)}dx\\
\le& C\int_\Omega |\Delta f|^2 e^{2s\varphi_{d_1}(x,0)}dx + Cs\int_\Omega |\nabla f|^2 e^{2s\varphi_{d_1}(x,0)}dx\\
+& C\max_{T-2\delta_1\le t\le T-\delta_1}\max_{x\in\overline{\Omega}} e^{2s\varphi_{d_1}(x,t)}(\|\Delta f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{\{L^2(\Omega)\}^n})\\
+& Ce^{Cs}D^2
\end{aligned}$$

for all $s\ge s_*$. The first term on the right-hand side can be absorbed into the left-hand side by choosing $s>0$ large, and by (4.13) we have

$$\int_\Omega s|\Delta f|^2 e^{2s\varphi_{d_1}(x,0)}dx$$

$$\le Cs\int_\Omega |\nabla f|^2 e^{2s\varphi_{d_1}(x,0)}dx + Ce^{2s(\mu_0 - r_0)}(\|\Delta f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{\{L^2(\Omega)\}^n}) + Ce^{Cs}D^2 \tag{4.17}$$

for all $s\ge s_*$.

Since $f = d_1 - d_2 \in H^2_0(\Omega)$ by $d_1, d_2\in\mathcal{D}$, we can apply (4.9) to obtain

$$\int_\Omega (s^2|\nabla f|^2 + s^4|f|^2)e^{2s\varphi_{d_1}(x,0)}dx \le C\int_\Omega s|\Delta f|^2 e^{2s\varphi_{d_1}(x,0)}dx.$$

Substituting this into the left-hand side of (4.17), we obtain

$$\begin{aligned}&\int_\Omega (s|\Delta f|^2 + s^2|\nabla f|^2 + s^4|f|^2)e^{2s\varphi_{d_1}(x,0)}dx\\ \le & Cs\int_\Omega |\nabla f|^2 e^{2s\varphi_{d_1}(x,0)}dx + Ce^{2s(\mu_0-r_0)}(\|\Delta f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{\{L^2(\Omega)\}^n}) + Ce^{Cs}D^2\end{aligned}$$

for all $s \ge s_*$, and absorbing the first term on the right-hand side into the left-hand side again by choosing $s > 0$ large if necessary, we have

$$\begin{aligned}& se^{2s\mu_0}(\|\Delta f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{\{L^2(\Omega)\}^n} + \|f\|^2_{L^2(\Omega)})\\ \le & s\int_\Omega (|\Delta f|^2 + |\nabla f|^2 + |f|^2)e^{2s\varphi_{d_1}(x,0)}dx\\ \le & Ce^{2s(\mu_0-r_0)}(\|\Delta f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{\{L^2(\Omega)\}^n}) + Ce^{Cs}D^2\end{aligned}$$

for all $s \ge s_*$. Noting $C^{-1}\|f\|^2_{H^2(\Omega)} \le \|\Delta f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{\{L^2(\Omega)\}^n} + \|f\|^2_{L^2(\Omega)} \le C\|f\|^2_{H^2(\Omega)}$, we see that

$$\|f\|_{H^2(\Omega)} \le Cs^{-1}e^{-2sr_0}\|f\|^2_{H^2(\Omega)} + Ce^{Cs}D^2$$

for all large $s > s_*$. Since $r_0 > 0$, for large $s > 0$, we can absorb the first term on the right-hand side into the left -hand side, so that

$$\|f\|^2_{H^2(\Omega)} \le 2Ce^{Cs}D^2.$$

Thus the proof of Theorem 3 is completed.

Appendix. Proof of Lemma 4

In order to clarify the dependence in the Carleman estimate on δ_0 and M, for arbitrary $d \in \mathcal{D}$, we prove the lemma directly and the proof is similar for example to Section 3 in Yamamoto [21], where a Carleman estimate is proved for a parabolic equation.

By the usual density argument, it suffices to prove for $f \in C_0^2(\Omega)$. We set

$$\psi_d(x) = \varphi_d(x,0), \quad g(x) = e^{s\psi_d(x)}f(x), \quad q = \Delta f$$

and

$$L_0 g(x) = e^{s\psi_d(x)}\Delta(e^{-s\psi_d(x)}g(x)).$$

We have

$$L_0 g = \Delta g - 2s\lambda\psi_d\nabla d\cdot\nabla g + s^2\lambda^2\psi_d^2|\nabla d|^2 g - s\lambda^2\psi_d|\nabla d|^2 g - s\lambda\psi_d(\Delta d)g \quad \text{in } \Omega. \tag{1}$$

We set

$$A_1 := -s\lambda^2\psi_d|\nabla d|^2 - s\lambda\psi_d(\Delta d). \tag{2}$$

We note that

$$|A_1(x,s,\lambda)| \le Cs\lambda^2\psi_d \quad \text{for } x \in \Omega \text{ and all large } \lambda > 0 \text{ and } s > 0.$$

Here and henceforth by C, C_1, etc., we denote generic constants which are independent of s, λ, ψ_d but dependent on $M, \delta_0 > 0$, and these constants may change line by line. Moreover we always assume that $\lambda \ge 1$. Hence we use the following inequality: $\lambda^m \le \lambda^{m'}$ if $0 < m \le m'$.

In view of A_1, we have

$$L_0 g = \Delta g - 2s\lambda\psi_d \nabla d \cdot \nabla g + s^2\lambda^2\psi_d^2|\nabla d|^2 g + A_1 g = q e^{s\psi_d} \quad \text{in } \Omega. \tag{3}$$

Taking into consideration the orders of (s, λ, ψ_d), we split $L_0 g$ as follows:

$$\begin{cases} L_1 g = \Delta g + s^2\lambda^2\psi_d^2|\nabla d|^2 g + A_1 g, \\ L_2 g = -2s\lambda\psi_d \nabla d \cdot \nabla g \quad \text{in } \Omega. \end{cases} \tag{4}$$

Since $\|qe^{s\psi_d}\|^2_{L^2(\Omega)} = \|L_1 g + L_2 g\|^2_{L^2(\Omega)}$, we obtain

$$2\int_\Omega (L_1 g)(L_2 g)dx \le \int_\Omega q^2 e^{2s\psi_d} dx. \tag{5}$$

Now we estimate

$$\begin{aligned} &\int_\Omega (L_1 g)(L_2 g)dx \\ = &-\int_\Omega 2s\lambda\psi_d(\nabla d\cdot\nabla g)\Delta g dx - \int_\Omega 2s^3\lambda^3\psi_d^3|\nabla d|^2(\nabla d\cdot\nabla g)g dx \end{aligned}$$

$$-\int_\Omega 2s\lambda\psi_d A_1(\nabla d\cdot\nabla g)g dx =: \sum_{j=1}^3 J_k. \tag{6}$$

By the integration by parts and $g = \partial_\nu g = 0$ on $\partial\Omega$, we have

$$\begin{aligned} J_1 = &\int_\Omega 2s\lambda\psi_d \sum_{j,k=1}^n (\partial_k d)(\partial_j\partial_k g)\partial_j g dx + \int_\Omega 2s\lambda \sum_{j,k=1}^n \partial_j((\partial_k d)\psi_d)(\partial_j g)\partial_k g dx \\ = &\int_\Omega s\lambda\psi_d \sum_{j,k=1}^n (\partial_k d)\partial_k(|\partial_j g|^2)dx + \int_\Omega 2s\lambda \sum_{j,k=1}^n \partial_j((\partial_k d)\psi_d)(\partial_j g)\partial_k g dx \\ = &-\int_\Omega s\lambda \sum_{k=1}^n \partial_k((\partial_k d)\psi_d)|\nabla g|^2 dx + \int_\Omega 2s\lambda \sum_{j,k=1}^n \partial_j((\partial_k d)\psi_d)(\partial_j g)\partial_k g dx. \end{aligned}$$

Since

$$\partial_j((\partial_k d)\psi_d) = (\partial_j\partial_k d)\psi_d + (\partial_k d)\partial_j\psi_d = \lambda(\partial_j d)(\partial_k d)\psi_d + (\partial_j\partial_k d)\psi_d,$$

we have

$$
\begin{aligned}
J_1 =& -\int_\Omega s\lambda^2\psi_d|\nabla d|^2|\nabla g|^2dx - \int_\Omega s\lambda(\Delta d)\psi_d|\nabla g|^2dx\\
&+\int_\Omega 2s\lambda^2\psi_d\sum_{j,k=1}^n(\partial_j d)(\partial_k d)(\partial_j g)\partial_k g dx + \int_\Omega 2s\lambda\psi_d\sum_{j,k=1}^n(\partial_j\partial_k d)(\partial_j g)\partial_k g dx\\
=& -\int_\Omega s\lambda^2\psi_d|\nabla d|^2|\nabla g|^2dx - \int_\Omega s\lambda(\Delta d)\psi_d|\nabla g|^2dx\\
&+\int_\Omega 2s\lambda^2\psi_d\left|\sum_{j,k=1}^n(\partial_j d)\partial_j g\right|^2dx + \int_\Omega 2s\lambda\psi_d\sum_{j,k=1}^n(\partial_j\partial_k d)(\partial_j g)\partial_k g dx\\
\ge& -\int_\Omega s\lambda^2\psi_d|\nabla d|^2|\nabla g|^2dx - \int_\Omega s\lambda(\Delta d)\psi_d|\nabla g|^2dx\\
&+\int_\Omega 2s\lambda\psi_d\sum_{j,k=1}^n(\partial_j\partial_k d)(\partial_j g)\partial_k g dx.
\end{aligned}
$$

Therefore

$$
J_1 \ge -\int_\Omega s\lambda^2\psi_d|\nabla d|^2|\nabla g|^2dx - C\int_\Omega s\lambda\psi_d|\nabla g|^2dx. \tag{7}
$$

Moreover we have

$$
\begin{aligned}
J_2 =& -\int_\Omega s^3\lambda^3\psi_d^3|\nabla d|^2\sum_{k=1}^n(\partial_k d)\partial_k(|g|^2)dx\\
=& \int_\Omega s^3\lambda^3\sum_{k=1}^n\partial_k(\psi_d^3)|\nabla d|^2(\partial_k d)|g|^2dx + \int_\Omega s^3\lambda^3\psi_d^3\sum_{k=1}^n\partial_k(|\nabla d|^2\partial_k d)|g|^2dx\\
=& \int_\Omega 3s^3\lambda^4\psi_d^3|\nabla d|^4|g|^2dx - C\int_\Omega s^3\lambda^3\psi_d^3|g|^2dx
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
J_3 =& -\int_\Omega s\lambda\psi_d A_1\sum_{k=1}^n(\partial_k d)\partial_k(|g|^2)dx\\
=& \int_\Omega\sum_{k=1}^n s\lambda\partial_k(\psi_d A_1)(\partial_k d)|g|^2dx + \int_\Omega\sum_{k=1}^n s\lambda\psi_d A_1(\partial_k^2 d)|g|^2dx\\
\ge& -C\int_\Omega s^2\lambda^4\psi_d^2|g|^2dx.
\end{aligned} \tag{9}
$$

Therefore (6) - (9) yield

$$
\begin{aligned}
\frac{1}{2}\int_\Omega |q|^2e^{2s\psi_d}dx \ge& \int_\Omega (L_1g)(L_2g)dx\\
\ge& -\int_\Omega s\lambda^2\psi_d|\nabla d|^2|\nabla g|^2dx + \int_\Omega 3s^3\lambda^4\psi_d^3|\nabla d|^4|g|^2dx
\end{aligned}
$$

$$-C\int_\Omega s\lambda\psi_d|\nabla g|^2dx - C\int_\Omega (s^3\lambda^3\psi_d^3 + s^2\lambda^4\psi_d^2)|g|^2dx. \tag{10}$$

For the proof of the lemma, we have to estimate $s\lambda^2\psi_d|\nabla g|^2 + s^3\lambda^4\psi_d^3|g|^2$, but the first and the second terms on the right-hand side have different signs. Thus we need another estimate. That is, multiplying (3) with $-s\lambda^2\psi_d|\nabla d|^2 g$ and integrating over Ω, we obtain

$$\sum_{j=1}^4 I_k := -\int_\Omega (\Delta g)s\lambda^2\psi_d|\nabla d|^2 gdx + \int_\Omega 2s^2\lambda^3\psi_d^2|\nabla d|^2(\nabla d\cdot\nabla g)gdx$$
$$-\int_\Omega s^3\lambda^4\psi_d^3|\nabla d|^4|g|^2dx - \int_\Omega s\lambda^2\psi_d A_1|\nabla d|^2|g|^2dx$$

$$= -\int_\Omega qe^{s\psi_d}s\lambda^2\psi_d|\nabla d|^2gdx. \tag{11}$$

Hence, by integration by parts, we obtain

$$I_1 = \int_\Omega s\lambda^2\psi_d|\nabla d|^2|\nabla g|^2dx + \int_\Omega s\lambda^2\nabla(|\nabla d|^2\psi_d)\cdot(\nabla g)gdx.$$

Here we have

$$\int_\Omega s\lambda^2\nabla(|\nabla d|^2\psi_d)\cdot(g\nabla g)dx \le C\int_\Omega s\lambda^3\psi_d|g||\nabla g|dx$$
$$=C\int_\Omega (s\lambda^2\psi_d|g|)(\lambda|\nabla g|)dx \le \frac{C}{2}\int_\Omega (s^2\lambda^4\psi_d^2|g|^2 + \lambda^2|\nabla g|^2)dx.$$

Consequently

$$I_1 \ge \int_\Omega s\lambda^2\psi_d|\nabla d|^2|\nabla g|^2dx - C\int_\Omega s^2\lambda^4\psi_d^2|g|^2dx - C\int_\Omega \lambda^2|\nabla g|^2dx. \tag{12}$$

Next

$$I_2 = \int_\Omega s^2\lambda^3\psi_d^2|\nabla d|^2(\nabla d\cdot\nabla(|g|^2))dx$$
$$= -\int_\Omega s^2\lambda^3\sum_{k=1}^n \partial_k(\psi_d^2)|\nabla d|^2(\partial_k d)|g|^2dx - \int_\Omega s^2\lambda^3\psi_d^2\sum_{k=1}^n \partial_k((\partial_k d)|\nabla d|^2)|g|^2dx$$

$$\ge -C\int_\Omega (s^2\lambda^4\psi_d^2 + s^2\lambda^3\psi_d^2)|g|^2dx \ge -C\int_\Omega s^2\lambda^4\psi_d^2|g|^2dx. \tag{13}$$

By (2), we see

$$I_4 \ge -C\int_\Omega s^2\lambda^4\psi_d^2|g|^2dx. \tag{14}$$

Since

$$\left|\int_\Omega qe^{s\psi_d}s\lambda^2\psi_d|\nabla d|^2gdx\right| \le \frac{1}{2}\int_\Omega |q|^2e^{2s\psi_d}dx + \frac{1}{2}\int_\Omega s^2\lambda^4\psi_d^2|\nabla d|^4|g|^2dx,$$

it follows from (11) - (14) that

$$\int_\Omega s\lambda^2\psi_d|\nabla d|^2|\nabla g|^2dx - \int_\Omega s^3\lambda^4\psi_d^3|\nabla d|^4|g|^2dx - C\int_\Omega s^2\lambda^4\psi_d^2|g|^2dx - C\int_\Omega \lambda^2|\nabla g|^2dx$$
$$\le\frac{1}{2}\int_\Omega |q|^2e^{2s\psi_d}dx + \frac{1}{2}\int_\Omega s^2\lambda^4\psi_d^2|\nabla d|^4|g|^2dx.$$

Therefore

$$\int_\Omega s\lambda^2\psi_d|\nabla d|^2|\nabla g|^2dx - \int_\Omega s^3\lambda^4\psi_d^3|\nabla d|^4|g|^2dx$$
$$\le \frac{1}{2}\int_\Omega |q|^2e^{2s\psi_d}dx + C\int_\Omega s^2\lambda^4\psi_d^2|g|^2dx + C\int_\Omega \lambda^2|\nabla g|^2dx. \tag{15}$$

Thus we consider (10) + 2× (15):

$$\int_\Omega s\lambda^2\psi_d|\nabla d|^2|\nabla g|^2dx + \int_\Omega s^3\lambda^4\psi_d^3|\nabla d|^4|g|^2dx$$
$$\le \frac{3}{2}\int_\Omega |q|^2e^{2s\psi_d}dx + C\int_\Omega (s\lambda\psi_d + \lambda^2)|\nabla g|^2dx + C\int_\Omega (s^3\lambda^3\psi_d^3 + s^2\lambda^4\psi_d^2)|g|^2dx. \tag{16}$$

Since $|\psi_d(x)| \ge e^{-\lambda M}$ for all $x \in \overline{\Omega}$ and $d \in \mathcal{D}$, we have

$$C\lambda^2 = s\lambda^2\psi_d \times \frac{C\psi_d^{-1}}{s} \le s\lambda^2\psi_d\frac{Ce^{\lambda M}}{s}, \tag{17}$$

$$Cs\lambda\psi_d = s\lambda^2\psi_d\frac{C}{\lambda}, \quad Cs^3\lambda^3\psi_d^3 = s^3\lambda^4\psi_d^3\frac{C}{\lambda} \tag{18}$$

and

$$Cs^2\lambda^4\psi_d^2 = s^3\lambda^4\psi_d^3\frac{C\psi_d^{-1}}{s} \le s^3\lambda^4\psi_d^3\frac{Ce^{\lambda M}}{s}. \tag{19}$$

We choose sufficiently large $\lambda_0 = \lambda_0(\delta_0, M) > 0$ such that $\frac{C}{\lambda_0} \le \frac{1}{4}$ in (18). For any given $\lambda \ge \lambda_0$, we choose $s_1 = s_1(\lambda, \delta_0, M)$ such that $\frac{Ce^{\lambda M}}{s_1} \le \frac{1}{4}$ in (17) and (19). Then

$$C\lambda^2, \quad Cs\lambda\psi_d \le \frac{1}{4}s\lambda^2\psi_d,$$
$$Cs^3\lambda^3\psi_d^3, \quad Cs^2\lambda^4\psi_d^2 \le \frac{1}{4}s^3\lambda^4\psi_d^3$$

for all $\lambda \ge \lambda_0$ and $s \ge s_1$, and noting $|\nabla d| \ge \delta_0 > 0$ for all $d \in \mathcal{D}$, we can absorb the second and the third terms on the right-hand side of (16) into the left-hand side to obtain

$$\frac{1}{2}\int_\Omega s\lambda^2\psi_d|\nabla g|^2dx + \frac{1}{2}\int_\Omega s^3\lambda^4\psi_d^3|g|^2dx \le C\int_\Omega |q|^2e^{2s\psi_d}dx. \tag{20}$$

We re-write by means of f. Replacing $g = e^{s\psi_d}f$, we have $|g|^2 = |f|^2e^{2s\psi_d}$ and

$$|\nabla f|^2e^{2s\psi_d} = |\nabla g - s\lambda(\nabla d)\psi_d e^{s\psi_d}f|^2$$
$$\le 2|\nabla g|^2 + 2s^2\lambda^2|\nabla d|^2\psi_d^2|f|^2e^{2s\psi_d} \le 2|\nabla g|^2 + Cs^2\lambda^2\psi_d^2|f|^2e^{2s\psi_d}.$$

Substituting them into (20), we complete the proof of Lemma 4.

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